

THE DERIVATIONS, CENTRAL EXTENSIONS AND AUTOMORPHISM GROUP OF THE LIE ALGEBRA W^*

SHOULAN GAO, CUIPO JIANG[†], AND YUFENG PEI

ABSTRACT. In this paper, we study the derivations, central extensions and the automorphisms of the infinite-dimensional Lie algebra W which appeared in [8] and Dong-Zhang's recent work [22] on the classification of some simple vertex operator algebras.

1. Introduction

It is well known that the Virasoro algebra Vir plays an important role in many areas of mathematics and physics (see [13], for example). It can be regarded as the universal central extension of the complexification of the Lie algebra $\text{Vect}(S^1)$ of (real) vector fields on the circle S^1 :

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c, \quad (1.1)$$

where c is a central element such that $[L_n, c] = 0$. The Virasoro algebra admits many interesting extensions and generalizations, for example, the W_N -algebras [21], $W_{1+\infty}$ [14], the higher rank Virasoro algebra [17], and the twisted Heisenberg-Virasoro algebra [1, 3, 11] etc. Recently M. Henke et al. [8, 9] investigated a Lie algebra W in their study of ageing phenomena which occur widely in physics [7]. The Lie algebra W is an abelian extension of centerless Virasoro algebra, and is isomorphic to the semi-direct product Lie algebra $\mathcal{L} \ltimes \mathcal{I}$, where \mathcal{L} is the centerless Virasoro algebra (Witt algebra) and \mathcal{I} is the adjoint \mathcal{L} -module. In particular, W is the infinite-dimensional extensions of the Poincare algebra \mathfrak{p}_3 . In their classification of the simple vertex operator algebras with 2 generators, Dong and Zhang [22] studied a similar infinite-dimensional Lie algebra $W(2, 2)$ and its representation theory. Although the algebra $W(2, 2)$ is an extension of the Virasoro algebra, as they remarked, the representation theory for $W(2, 2)$ is totally different from that of the Virasoro algebra.

The purpose of this paper is to study the structure of the Lie algebra W and its universal central extension \widetilde{W} . We will see that there is a natural surjective homomorphism from the universal covering algebra \widetilde{W} to $W(2, 2)$. We show that the second cohomology group with trivial coefficients for the Lie algebra W is two dimensional.

Keywords: derivation, central extension, automorphism.

* Supported in part by NSFC grant 10571119 and NSF grant 06ZR14049 of Shanghai City.

[†] Corresponding author: cpjiang@sjtu.edu.cn.

Furthermore, by the Hochschild-Serre spectral sequence [20] and R. Farnsteiner's theorem [5], we determine the derivation algebras of W and \widetilde{W} , which both have only one outer derivation. Finally, the automorphism groups of W and \widetilde{W} are also characterized.

Throughout the paper, we denote by \mathbb{Z} the set of all integers and \mathbb{C} the field of complex numbers.

2. The Universal Central Extension of W

The Lie algebra W over the complex field \mathbb{C} has a basis $\{L_m, I_m \mid m \in \mathbb{Z}\}$ with the following bracket

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, I_n] = (m - n)I_{m+n}, \quad [I_m, I_n] = 0,$$

for all $m, n \in \mathbb{Z}$. It is clear that W is isomorphic to the semi-direct product Lie algebra $W \simeq \mathcal{L} \ltimes \mathcal{I}$, where $\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n$ is the classical Witt algebra and $\mathcal{I} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}I_n$ can be regarded as the adjoint \mathcal{L} -module. Moreover, $W = \bigoplus_{m \in \mathbb{Z}} W_m$ is a \mathbb{Z} -graded Lie algebra, where $W_m = \mathbb{C}L_m \oplus \mathbb{C}I_m$.

Let \mathfrak{g} be a Lie algebra. Recall that a bilinear function $\psi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ is called a 2-cocycle on \mathfrak{g} if for all $x, y, z \in \mathfrak{g}$, the following two conditions are satisfied:

$$\begin{aligned} \psi(x, y) &= -\psi(y, x), \\ \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) &= 0. \end{aligned} \tag{2.1}$$

For any linear function $f : \mathfrak{g} \longrightarrow \mathbb{C}$, one can define a 2-cocycle ψ_f as follows

$$\psi_f(x, y) = f([x, y]), \quad \forall x, y \in \mathfrak{g}.$$

Such a 2-cocycle is called a 2-coboundary on \mathfrak{g} .

Let \mathfrak{g} be a perfect Lie algebra, i.e., $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Denote by $C^2(\mathfrak{g}, \mathbb{C})$ the vector space of 2-cocycles on \mathfrak{g} , $B^2(\mathfrak{g}, \mathbb{C})$ the vector space of 2-coboundaries on \mathfrak{g} . The quotient space:

$$H^2(\mathfrak{g}, \mathbb{C}) = C^2(\mathfrak{g}, \mathbb{C}) / B^2(\mathfrak{g}, \mathbb{C})$$

is called the second cohomology group of \mathfrak{g} with trivial coefficients \mathbb{C} . It is well-known that $H^2(\mathfrak{g}, \mathbb{C})$ is one-to-one correspondence to the equivalence classes of one-dimensional central extensions of the Lie algebra \mathfrak{g} .

We will determine the second cohomology group for the Lie algebra W .

Lemma 2.1 (See also [16]). *Let $(\mathfrak{g}, [\cdot, \cdot]_0)$ be a perfect Lie algebra over \mathbb{C} and V a \mathfrak{g} -module such that $\mathfrak{g} \cdot V = V$. Consider the semi-direct product Lie algebra $(\mathfrak{g} \ltimes V, [\cdot, \cdot])$ with the following bracket*

$$[x, y] = [x, y]_0, \quad [x, v] = x \cdot v, \quad [u, v] = 0, \quad \forall x, y \in \mathfrak{g}, \quad u, v \in V.$$

Let V^ be the dual \mathfrak{g} -module and*

$$B^{\mathfrak{g}}(V) = \{f \in \text{Hom}(V \otimes V, \mathbb{C}) \mid f(u, v) = -f(v, u), \quad f(x \cdot u, v) + f(u, x \cdot v) = 0, \quad \forall x \in \mathfrak{g}, u, v \in V\}.$$

Then we have

$$H^2(\mathfrak{g} \ltimes V, \mathbb{C}) = H^2(\mathfrak{g}, \mathbb{C}) \oplus H^1(\mathfrak{g}, V^*) \oplus B^{\mathfrak{g}}(V).$$

Proof. Let α be a 2-cocycle on $\mathfrak{g} \ltimes V$. Obviously, $\alpha|_{\mathfrak{g}} \in H^2(\mathfrak{g}, \mathbb{C})$. Define $D_\alpha \in \text{Hom}_{\mathbb{C}}(\mathfrak{g}, V^*)$ by

$$D_\alpha(x)(v) = \alpha(x, v),$$

for $x \in \mathfrak{g}, v \in V$. Then $D_\alpha \in H^1(\mathfrak{g}, V^*)$. In fact, for any $x, y \in \mathfrak{g}, v \in V$,

$$\begin{aligned} \alpha([x, y], v) &= \alpha(x \cdot v, y) - \alpha(y \cdot v, x) \\ &= -D_\alpha(y)(x \cdot v) + D_\alpha(x)(y \cdot v) \\ &= (x \cdot D_\alpha(y))(v) - (y \cdot D_\alpha(x))(v). \end{aligned}$$

Therefore

$$D_\alpha([x, y]) = x \cdot D_\alpha(y) - y \cdot D_\alpha(x),$$

for all $x, y \in \mathfrak{g}$. Define $f_\alpha(u, v) = \alpha(u, v)$ for any $u, v \in V$, then $f_\alpha \in B^{\mathfrak{g}}(V)$. It is straightforward to check $\alpha|_{\mathfrak{g}}, D_\alpha$ and f_α are linearly independent. We get the desired formula. \square

Corollary 2.2. *Let $V = \mathfrak{g}$ be the adjoint \mathfrak{g} -module. Then*

$$H^2(\mathfrak{g} \ltimes \mathfrak{g}, \mathbb{C}) = H^2(\mathfrak{g}, \mathbb{C}) \oplus H^1(\mathfrak{g}, \mathfrak{g}^*) \oplus B^{\mathfrak{g}}(\mathfrak{g}). \quad (2.2)$$

Theorem 2.3. $H^2(W, \mathbb{C}) = \mathbb{C}\alpha \oplus \mathbb{C}\beta$, where

$$\begin{aligned} \alpha(L_m, L_n) &= \delta_{m+n,0} \frac{m^3 - m}{12}, \quad \alpha(L_m, I_n) = \alpha(I_m, I_n) = 0, \\ \beta(L_m, I_n) &= \delta_{m+n,0} \frac{m^3 - m}{12}, \quad \beta(L_m, L_n) = \beta(I_m, I_n) = 0. \end{aligned}$$

for any $m, n \in \mathbb{Z}$.

Proof. Since $H^2(\mathcal{L}, \mathbb{C}) = \mathbb{C}\alpha$ and $H^1(\mathcal{L}, \mathcal{L}^*) \simeq \mathbb{C}\beta$ (see [6, 10] for the details), we need to prove that $B^{\mathcal{L}}(\mathcal{I}) = 0$. Let $f \in B^{\mathcal{L}}(\mathcal{I})$, then

$$(i - j)f(I_{i+j}, I_k) + (k - i)f(I_{k+i}, I_j) = 0. \quad (2.3)$$

Letting $i = 0$ in (2.3), we get

$$(j + k)f(I_j, I_k) = 0.$$

So $f(I_j, I_k) = 0$ for $j + k \neq 0$. Let $k = -i - j$ in (2.3), then we obtain

$$(i - j)f(I_{i+j}, I_{-i-j}) + (2i + j)f(I_j, I_{-j}) = 0. \quad (2.4)$$

Let $j = -i$, then $2if(I_0, I_0) = if(I_i, I_{-i})$, which implies that $f(I_i, I_{-i}) = 0$ for all $i \in \mathbb{Z}$. Therefore, $f(I_m, I_n) = 0$ for all $m, n \in \mathbb{Z}$. \square

Let \mathfrak{g} be a Lie algebra, $(\widetilde{\mathfrak{g}}, \pi)$ is called a central extension of \mathfrak{g} if $\pi : \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is a surjective homomorphism whose kernel lies in the center of the Lie algebra $\widetilde{\mathfrak{g}}$. The pair $(\widetilde{\mathfrak{g}}, \pi)$ is called a covering of \mathfrak{g} if $\widetilde{\mathfrak{g}}$ is perfect. A covering $(\widetilde{\mathfrak{g}}, \pi)$ is called a universal central extension of \mathfrak{g} if for every central extension $(\widetilde{\mathfrak{g}}', \varphi)$ of \mathfrak{g} there is a unique homomorphism $\psi : \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}}'$ for which $\varphi\psi = \pi$. It follows from [7] that every perfect Lie algebra has a universal central extension.

Let $\widetilde{W} = W \oplus \mathbb{C} C_1 \oplus \mathbb{C} C_2$ be a vector space over the complex field \mathbb{C} with a basis $\{L_n, I_n, C_1, C_2 \mid n \in \mathbb{Z}\}$ satisfying the following relations

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C_1,$$

$$[L_m, I_n] = (m - n)I_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C_2,$$

$$[I_m, I_n] = 0, \quad [C_1, \widetilde{W}] = 0, \quad [C_2, \widetilde{W}] = 0,$$

for all $m, n \in \mathbb{Z}$. By Theorem 2.3, \widetilde{W} is a universal covering algebra of W .

There is a \mathbb{Z} -grading on \widetilde{W} :

$$\widetilde{W} = \bigoplus_{n \in \mathbb{Z}} \widetilde{W}_n,$$

where $\widetilde{W}_n = \text{span}\{L_n, I_n, \delta_{n,0}C_1, \delta_{n,0}C_2\}$. Set $\widetilde{W}_+ = \bigoplus_{n>0} \widetilde{W}_n$ and $\widetilde{W}_- = \bigoplus_{n<0} \widetilde{W}_n$, then there is a triangular decomposition on \widetilde{W} :

$$\widetilde{W} = \widetilde{W}_- \oplus \widetilde{W}_0 \oplus \widetilde{W}_+.$$

Remark 2.4. Let $C_1 = C_2 = C$, then we get the Lie algebra $W(2, 2)$ appeared in [22].

3. The Derivation Algebra of W

In this section, we consider the derivation algebra of W .

Let G be a commutative group, $\mathfrak{g} = \bigoplus_{g \in G} \mathfrak{g}_g$ a G -graded Lie algebra. A \mathfrak{g} -module V is called G -graded, if

$$V = \bigoplus_{g \in G} V_{g \in G}, \quad \mathfrak{g}_g V_h \subseteq V_{g+h}, \quad \forall g, h \in G.$$

Let \mathfrak{g} be a Lie algebra and V a \mathfrak{g} -module. A linear map $D : \mathfrak{g} \longrightarrow V$ is called a derivation, if for any $x, y \in \mathfrak{g}$,

$$D[x, y] = x.D(y) - y.D(x).$$

If there exists some $v \in V$ such that $D : x \mapsto x.v$, then D is called an inner derivation. Denote by $\text{Der}(\mathfrak{g}, V)$ the vector space of all derivations, $\text{Inn}(\mathfrak{g}, V)$ the vector space of all inner derivations. Set

$$H^1(\mathfrak{g}, V) = \text{Der}(\mathfrak{g}, V) / \text{Inn}(\mathfrak{g}, V).$$

Denote by $Der(\mathfrak{g})$ the derivation algebra of \mathfrak{g} , $Inn(\mathfrak{g})$ the vector space of all inner derivations of \mathfrak{g} .

Firstly, we have the short exact sequence of Lie algebras

$$0 \rightarrow \mathcal{I} \rightarrow W \rightarrow \mathcal{L} \rightarrow 0,$$

which induces an exact sequence

$$H^1(W, \mathcal{I}) \rightarrow H^1(W, W) \rightarrow H^1(W, \mathcal{L}). \quad (3.1)$$

The right-hand side can be computed from the initial terms

$$0 \rightarrow H^1(\mathcal{L}, \mathcal{L}) \rightarrow H^1(W, \mathcal{L}) \rightarrow H^1(\mathcal{I}, \mathcal{L})^W \quad (3.2)$$

of the four-term sequence associated to the Hochschild-Serre spectral sequence [20]. It is known that $H^1(\mathcal{L}, \mathcal{L}) = 0$, while the term $H^1(W, \mathcal{L})^W$ is equal to $\text{Hom}_{U(W)}(\mathcal{I}/[\mathcal{I}, \mathcal{I}], \mathcal{L})$. Therefore, it suffices to compute the terms $H^1(W, \mathcal{I})$ and $\text{Hom}_{U(W)}(\mathcal{I}/[\mathcal{I}, \mathcal{I}], \mathcal{L})$.

By Proposition 1.1 in [5], we have the following lemma.

Lemma 3.1.

$$Der(W, \mathcal{I}) = \bigoplus_{n \in \mathbb{Z}} Der(W, \mathcal{I})_n,$$

where $Der(W, \mathcal{I})_n(\mathcal{I}_m) \subseteq \mathcal{I}_{m+n}$ for all $m, n \in \mathbb{Z}$.

□

Lemma 3.2. *For any nonzero integer m , we have*

$$H^1(W_0, \mathcal{I}_m) = 0,$$

where $\mathcal{I}_m = \mathbb{C}I_m$.

Proof. Let $m \neq 0$, $\varphi : W_0 \rightarrow \mathbb{C}I_m$ a derivation. Assume that

$$\varphi(L_0) = aI_m, \quad \varphi(I_0) = bI_m,$$

where $a, b \in \mathbb{C}$. Since $\varphi[L_0, I_0] = [\varphi(L_0), I_0] + [L_0, \varphi(I_0)]$, it is easy to deduce $b = 0$. Let $E_m = -\frac{a}{m}I_m$, then we have $\varphi(L_0) = [L_0, E_m]$ and $\varphi(I_0) = [I_0, E_m]$. Therefore, $\varphi \in Inn(W_0, I_m)$, that is,

$$H^1(W_0, I_m) = 0, \quad \forall m \in \mathbb{Z} \setminus \{0\}.$$

□

Lemma 3.3. $\text{Hom}_{W_0}(W_m, \mathcal{I}_n) = 0$ for all $m, n \in \mathbb{Z}$, $m \neq n$.

Proof. Let $f \in \text{Hom}_{W_0}(W_m, \mathcal{I}_n)$, where $m \neq n$. Then for any $E_m \in W_m$, we have

$$f([L_0, E_m]) = [L_0, f(E_m)].$$

Note $[L_0, E_k] = -kE_k$ for all $k \in \mathbb{Z}$. Then we get

$$-mf(E_m) = [L_0, f(E_m)] = -nf(E_m).$$

So $f(E_m) = 0$ for all $m \neq n$. Therefore, we have $f = 0$.

□

By Lemma 4.3-4.4 and Proposition 1.2 in [5], we have the following Lemma.

Lemma 3.4. $Der(W, \mathcal{I}) = Der(W, \mathcal{I})_0 + Inn(W, \mathcal{I})$.

Lemma 3.5. $H^1(W, \mathcal{I}) = \mathbb{C}D_1$, where

$$D_1(L_m) = 0, \quad D_1(I_m) = I_m,$$

for all $m \in \mathbb{Z}$.

Proof. For any $D \in Der(W, \mathcal{I})_0$, assume

$$D(L_m) = a_m I_m, \quad D(I_m) = b_m I_m,$$

where $a_m, b_m \in \mathbb{C}$. By the definition of derivation and the Lie bracket in W , we have

$$a_{m+n} = a_m + a_n, \quad b_{m+n} = b_n, \quad m \neq n.$$

Obviously, $b_m = b_0$ for all $m \in \mathbb{Z}$ and $a_0 = 0, a_{-m} = -a_m$. By induction on $m > 0$, we deduce that $a_m = (m-2)a_1 + a_2$ for $m > 2$. Then it is easy to infer that $a_2 = 2a_1$. Consequently, we get $a_m = ma_1$ for all $m \in \mathbb{Z}$. So for all $m \in \mathbb{Z}$, we have

$$D(L_m) = ma_1 I_m, \quad D(I_m) = b_1 I_m.$$

Set $D_0 = -ad(a_1 I_0) \in Inn(W)$, then $D(L_m) = D_0(L_m)$, $D(I_m) = D_0(I_m) + b_1 I_m$. Therefore,

$$\bar{D}(L_m) = 0, \quad \bar{D}(I_m) = b_1 I_m,$$

for all $m \in \mathbb{Z}$, where $\bar{D} = D - D_0$ is an outer derivation. The lemma holds. \square

Lemma 3.6. $\text{Hom}_{U(W)}(\mathcal{I}/[\mathcal{I}, \mathcal{I}], \mathcal{L}) = 0$.

Proof. As a matter of fact,

$$\text{Hom}_{U(W)}(\mathcal{I}/[\mathcal{I}, \mathcal{I}], \mathcal{L}) = \text{Hom}_{U(W)}(\mathcal{I}, \mathcal{L}).$$

For any $f \in \text{Hom}_{U(W)}(\mathcal{I}, \mathcal{L})$, we have $[L_0, f(I_m)] = f([L_0, I_m])$, i.e.,

$$ad(-L_0)(f(I_m)) = mf(I_m),$$

for all $m \in \mathbb{Z}$. This suggests $f(I_m) \in \mathbb{C}L_m$. Assume $f(I_m) = x_m L_m$ for all $m \in \mathbb{Z}$, where $x_m \in \mathbb{C}$. By the relation that $[L_n, f(I_m)] = f([L_n, I_m])$ for all $m, n \in \mathbb{Z}$, we have

$$x_{m+n} = x_m, \quad m \neq n.$$

Obviously, $x_m = x_0$ for all $m \in \mathbb{Z}$. So there exists some constant $a \in \mathbb{C}$ such that

$$f(I_m) = aL_m,$$

for all $m \in \mathbb{Z}$. Since $f([I_0, I_1]) = 0 = [I_0, f(I_1)] = -aL_1$, we have $a = 0$. Hence $f = 0$. \square

Theorem 3.7. $H^1(W, W) = \mathbb{C}D$, where

$$D(L_m) = 0, \quad D(I_m) = I_m,$$

for all $m \in \mathbb{Z}$.

Furthermore, it follows from Theorem 2.2 in [2] that

Corollary 3.8. $H^1(\widetilde{W}, \widetilde{W}) \simeq H^1(W, W)$.

4. The Automorphism Group of W

Denote by $Aut(W)$ and \mathfrak{Z} the automorphism group and the inner automorphism group of W respectively. Obviously, \mathfrak{Z} is generated by $\exp(kadI_m)$, $m \in \mathbb{Z}$, $k \in \mathbb{C}$, and \mathfrak{Z} is an abelian subgroup. Note that \mathcal{I} is the maximal proper ideal of W , so we have the following lemma.

Lemma 4.1. For any $\sigma \in Aut(W)$, $\sigma(I_n) \in \mathcal{I}$ for all $n \in \mathbb{Z}$.

□

For any $\prod_{j=s}^t \exp(k_{i_j} adI_{i_j}) \in \mathfrak{Z}$, we have

$$\prod_{j=s}^t \exp(k_{i_j} adI_{i_j})(I_n) = I_n, \quad \prod_{j=s}^t \exp(k_{i_j} adI_{i_j})(L_n) = L_n + \sum_{j=s}^t k_{i_j}(i_j - n)I_{i_j+n}.$$

Lemma 4.2. For any $\sigma \in Aut(W)$, there exist some $\tau \in \mathfrak{Z}$ and $\epsilon \in \{\pm 1\}$ such that

$$\bar{\sigma}(L_n) = a^n \epsilon L_{\epsilon n} + a^n \lambda n I_{\epsilon n}, \quad (4.1)$$

$$\bar{\sigma}(I_n) = a^n \mu I_{\epsilon n}, \quad (4.2)$$

where $\bar{\sigma} = \tau^{-1}\sigma$, $a, \mu \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}$. Conversely, if $\bar{\sigma}$ is a linear operator on \mathfrak{sw} satisfying (4.1)-(4.2) for some $\epsilon \in \{\pm 1\}$, $a, \mu \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}$, then $\bar{\sigma} \in Aut(W)$.

Proof. For any $\sigma \in Aut(W)$, denote $\sigma|_{\mathcal{L}} = \sigma'$. Then σ' is an automorphism of the classical Witt algebra, so $\sigma'(L_m) = \epsilon a^m L_{\epsilon m}$ for all $m \in \mathbb{Z}$, where $a \in \mathbb{C}^*$ and $\epsilon \in \{\pm 1\}$. Assume that

$$\sigma(L_0) = \epsilon L_0 + \sum_{i=p}^q \lambda_i I_i + \lambda_0 I_0,$$

where $i \neq 0$. Let $\tau = \prod_{i=p}^q \exp(\frac{\lambda_i}{\epsilon i} adI_i)$, then

$$\tau(\epsilon L_0) = \epsilon L_0 + \sum_{i=p}^q \lambda_i I_i.$$

Therefore, $\sigma(L_0) = \tau(\epsilon L_0) + \lambda_0 I_0$. Set $\bar{\sigma} = \tau^{-1}\sigma$, then $\bar{\sigma}(L_0) = \epsilon L_0 + \lambda_0 I_0$. Assume

$$\bar{\sigma}(L_n) = a^n \epsilon L_{\epsilon n} + a^n \sum \lambda(n_i) I_{n_i}, \quad n \neq 0,$$

where each formula is of finite terms and $\lambda(n_i), \mu(n_j) \in \mathbb{C}$. For $m \neq 0$, through the relation $[\bar{\sigma}(L_0), \bar{\sigma}(L_m)] = -m\bar{\sigma}(L_m)$, we get

$$\lambda_0 m I_{\epsilon m} = \sum \lambda(m_i) [m - \epsilon m_i] I_{m_i}.$$

This forces that $\lambda_0 = 0$. Then $[\bar{\sigma}(L_0), \bar{\sigma}(L_m)] = [\epsilon L_0, \bar{\sigma}(L_m)] = -m\bar{\sigma}(L_m)$, that is,

$$-adL_0(\bar{\sigma}(L_m)) = \epsilon m \bar{\sigma}(L_m).$$

So for all $m \in \mathbb{Z}$, we obtain

$$\bar{\sigma}(L_m) = a^m \epsilon L_{\epsilon m} + a^m \lambda(\epsilon m) I_{\epsilon m}.$$

Comparing the coefficients of $I_{\epsilon(m+n)}$ on the both side of $[\bar{\sigma}(L_m), \bar{\sigma}(L_n)] = (m-n)\bar{\sigma}(L_{m+n})$, we get

$$\lambda(\epsilon m) + \lambda(\epsilon n) = \lambda(\epsilon(m+n)).$$

Note $\lambda_0 = 0$, we deduce that $\lambda(\epsilon m) = m\lambda(\epsilon)$ for all $m \in \mathbb{Z}$. Therefore,

$$\bar{\sigma}(L_m) = a^m \epsilon L_{\epsilon m} + a^m m \lambda(\epsilon) I_{\epsilon m}.$$

Finally, by $[\sigma(L_0), \sigma(I_m)] = -m\sigma(I_m)$, we have

$$-adL_0(\sigma(I_m)) = \epsilon m \sigma(I_m),$$

for all $m \in \mathbb{Z}$. Then by Lemma 4.1, we may assume

$$\sigma(I_n) = a^n \mu(\epsilon n) I_{\epsilon n},$$

for all $n \in \mathbb{Z}$. By $[\sigma(L_m), \sigma(I_n)] = (m-n)\sigma(I_{m+n})$, we get

$$\mu(\epsilon n) = \mu(\epsilon(m+n)), \quad m \neq n.$$

Consequently, $\mu(\epsilon m) = \mu(0)$ for all $m \in \mathbb{Z}$. Set $\mu(0) = \mu$, then for all $n \in \mathbb{Z}$, we have

$$\bar{\sigma}(I_n) = a^n \mu I_{\epsilon n}.$$

□

Denote by $\bar{\sigma}(\epsilon, \lambda, a, \mu)$ the automorphism of W satisfying (4.1)-(4.2), then

$$\bar{\sigma}(\epsilon_1, \lambda_1, a_1, \mu_1) \bar{\sigma}(\epsilon_2, \lambda_2, a_2, \mu_2) = \bar{\sigma}(\epsilon_1 \epsilon_2, \lambda_1 + \mu_1 \lambda_2, a_1^{\epsilon_2} a_2, \mu_1 \mu_2), \quad (4.3)$$

and $\bar{\sigma}(\epsilon_1, \lambda_1, a_1, \mu_1) = \bar{\sigma}(\epsilon_2, \lambda_2, a_2, \mu_2)$ if and only if $\epsilon_1 = \epsilon_2, \lambda_1 = \lambda_2, a_1 = a_2, \mu_1 = \mu_2$. Let

$$\bar{\pi}_\epsilon = \bar{\sigma}(\epsilon, 0, 1, 1), \quad \bar{\sigma}_\lambda = \bar{\sigma}(1, \lambda, 1, 1), \quad \bar{\sigma}_{a, \mu} = \bar{\sigma}(1, 0, a, \mu)$$

and

$$\mathbf{a} = \{\bar{\pi}_\epsilon \mid \epsilon = \pm 1\}, \quad \mathbf{t} = \{\bar{\sigma}_\lambda \mid \lambda \in \mathbb{C}\}, \quad \mathbf{b} = \{\bar{\sigma}_{a, \mu} \mid a, \mu \in \mathbb{C}^*\}.$$

By (4.3), we have the following relations:

$$\bar{\sigma}(\epsilon, \lambda, a, \mu) = \bar{\sigma}(\epsilon, 0, 1, 1) \bar{\sigma}(1, \lambda, 1, 1) \bar{\sigma}(1, 0, a, \mu) \in \mathbf{atb},$$

$$\bar{\sigma}(\epsilon, \lambda, a, \mu)^{-1} = \bar{\sigma}(\epsilon, -\lambda \mu^{-1}, a^{-\epsilon}, \mu^{-1}),$$

$$\bar{\pi}_{\epsilon_1} \bar{\pi}_{\epsilon_2} = \bar{\pi}_{\epsilon_1 \epsilon_2}, \quad \bar{\sigma}_{\lambda_1} \bar{\sigma}_{\lambda_2} = \bar{\sigma}_{\lambda_1 + \lambda_2}, \quad \bar{\sigma}_{a_1, \mu_1} \bar{\sigma}_{a_2, \mu_2} = \bar{\sigma}_{a_1 a_2, \mu_1 \mu_2},$$

$$\bar{\sigma}_\lambda \bar{\pi}_\epsilon = \bar{\pi}_\epsilon \bar{\sigma}_\lambda, \quad \bar{\pi}_\epsilon^{-1} \bar{\sigma}_{a,\mu} \bar{\pi}_\epsilon = \bar{\sigma}_{a^\epsilon, \mu}, \quad \bar{\sigma}_{a,\mu} \bar{\sigma}_\lambda \bar{\sigma}_{a,\mu}^{-1} = \bar{\sigma}_{\mu\lambda}.$$

Hence, the following lemma holds.

Lemma 4.3. $\mathfrak{a}, \mathfrak{t}$ and \mathfrak{b} are all subgroups of $\text{Aut}(W)$. Furthermore, \mathfrak{t} is an abelian normal subgroup commutative with \mathfrak{z} .

$$\text{Aut}(W) = (\mathfrak{z}\mathfrak{t}) \rtimes (\mathfrak{a} \ltimes \mathfrak{b}),$$

where $\mathfrak{a} \cong \mathbb{Z}_2 = \{\pm 1\}$, $\mathfrak{t} \cong \mathbb{C}$, $\mathfrak{b} \cong \mathbb{C}^* \times \mathbb{C}^*$. □

Let $\mathbb{C}^\infty = \{(a_i)_{i \in \mathbb{Z}} \mid a_i \in \mathbb{C}, \text{ all but a finite number of the } a_i \text{ are zero}\}$. Then \mathbb{C}^∞ is an abelian group.

Lemma 4.4. $\mathfrak{z}\mathfrak{t}$ is isomorphic to \mathbb{C}^∞ .

Proof. Define $f : \mathfrak{z}\mathfrak{t} \longrightarrow \mathbb{C}^\infty$ by

$$f(\bar{\sigma}_\lambda \prod_{i=1}^s \exp(\alpha_{k_i} \text{ad} I_{k_i})) = (a_p)_{p \in \mathbb{Z}},$$

where $a_{k_i} = \alpha_{k_i}$ for $k_i < 0$, $a_0 = \lambda$, $a_{k_i+1} = \alpha_{k_i}$ for $k_i \geq 0$, and the others are zero, $k_i \in \mathbb{Z}$ and $k_1 < k_2 < \dots < k_s$. Since every element of $\mathfrak{z}\mathfrak{t}$ has the unique form of $\bar{\sigma}_\lambda \prod_{i=1}^s \exp(\alpha_{k_i} \text{ad} I_{k_i})$, it is easy to check that f is an isomorphism of group. □

Theorem 4.5. $\text{Aut}(W) \cong \mathbb{C}^\infty \rtimes (\mathbb{Z}_2 \ltimes (\mathbb{C}^* \times \mathbb{C}^*))$. □

Since W is centerless, it follows from Corollary 6 in [18] that $\text{Aut}(\widetilde{W}) = \text{Aut}(W)$, that is,

$$\text{Aut}(\widetilde{W}) \cong \mathbb{C}^\infty \rtimes (\mathbb{Z}_2 \ltimes (\mathbb{C}^* \times \mathbb{C}^*)).$$

REFERENCES

- [1] E. Arbarello, C. De Concini, V.G. Kac, C. Procesi, *Moduli spaces of curves and representation theory*, Comm. Math. Phys. **117** (1988), 1-36.
- [2] G. M. Benkart, R. V. Moody, *Derivations, central extensions and affine Lie algebras*, Algebras Groups Geom. **3** (1986), 456-492.
- [3] Y. Billig, *Representations of the twisted Heisenberg-Virasoro algebra at level zero*, Canadian Mathematical Bulletin, **46** (2003), 529-537.
- [4] D. Ž. Doković, K. Zhao, *Derivations, isomorphisms, and second cohomology of generalized Witt algebras*, Tran. Ams. Math. Soc. **350**(2) (1998), 643-664.
- [5] R. Farnsteiner, *Derivations and extensions of finitely generated graded Lie algebras*, J. Algebra **118** (1) (1988), 34-45.
- [6] J. L. Loday, T. Pirashvili, *Universal enveloping algebras of Leibniz algebras and (co)-homology*, Math. Ann. **296** (1993), 138-158.
- [7] M. Henkel, *Ageing, dynamical scaling and its extensions in many-particle systems without detailed balance*, J. Phys. Cond. Matt. **19** (2007).

- [8] M. Henkel, R. Schott, S. Stoimenov, J. Unterberger, *The Poincare algebra in the context of ageing systems: Lie structure, representations, Appell systems and coherent states*, arXiv:math-ph/0601028 (2006).
- [9] M. Henkel, R. Schott, S. Stoimenov, J. Unterberger, *On the dynamical symmetric algebra of ageing: Lie structure, representations and Appell systems*, QP-PQ: Quantum probability and white-noise analysis **20** (2007), 233-240.
- [10] N. Hu, Y. Pei, D. Liu, *A cohomological characterization of Leibniz central extensions of Lie algebras*, Proc. Amer. Math. Soc. **136** (2008), 437-447.
- [11] Q. Jiang, C. Jiang, *Representations of the twisted Heisenberg-Virasoro algebra and the full toroidal Lie algebras*, Algebra Colloq. **14**(1) (2007), 117-134 .
- [12] C. Jiang, D. Meng, *The derivation algebra of the associative algebra $C_q[X, Y, X^{-1}, Y^{-1}]$* , Comm. Algebra **6** (1998), 1723-1736.
- [13] V. Kac, A. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Advanced Series in Mathematics **2** (1987).
- [14] V. Kac and A. Radul, *Quasi-finite highest weight modules over the Lie algebra of differential operators on the circle*, Comm. Math. Phys. **157** (1993), 429-457.
- [15] D. Liu, C. Jiang, *The generalized Heisenberg-Virasoro algebra*, arXiv:math/0510543v3 [math.RT]
- [16] V. Ovsienko, C. Roger, *Extensions of the Virasoro group and the Virasoro algebra by modules of tensor densities on S^1* , Func. Anal. Appl. **30** (1996), 1573-8485.
- [17] J. Patera, and H. Zassenhaus, *The higher rank Virasoro algebras*, Comm. Math. Phys. **136** (1991), 1-14.
- [18] A. Pianzola, *Automorphisms of toroidal Lie algebras and their central quotients*, J. Algebra Appl. **1** (2002), 113-121.
- [19] R. Shen, C. Jiang, *The derivation algebra and automorphism group of the twisted Heisenberg-Virasoro algebra*, Commun. Alg. **34** (7) (2006), 2547-2558.
- [20] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1994.
- [21] A. B. Zamolodchikov, *Infinite additional symmetries in two dimensional conformal quantum field theory*, Theor. Math. Phys. **65** (1985), 1205-1213.
- [22] W. Zhang, C. Dong, *W algebra $W(2, 2)$ and the vertex operator algebra $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$* , arXiv:0711.4624v1 (2007).

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAOTONG UNIVERSITY, SHANGHAI 200240, CHINA
E-mail address: gaoshoulan@sjtu.edu.cn

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAOTONG UNIVERSITY, SHANGHAI 200240, CHINA
E-mail address: cpjiang@sjtu.edu.cn

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAOTONG UNIVERSITY, SHANGHAI 200240, CHINA
E-mail address: yfpei@sjtu.edu.cn